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The flow and heat transfer between a torsionally oscillating and a stationary disk

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SUMMARY

The flow between a torsionally oscillating and a stationary disk is considered. The solution in the form of a power series in ϵ, the ratio of the amplitude of oscillations to the frequency, is obtained on the assumption that $\epsilon << 1$. The steady streaming motion is shown to be of the order of magnitude ϵ. The behaviour of steady radial and axial velocities is clarified. The shearing stress on either disk is also calculated.

The problem of heat transfer in this flow field is considered next. The time-averaged Nusselt numbers are calculated up to the third-order approximation in the ϵ-expansion. The effect of the steady streaming flow upon the time-averaged heat-transfer rate is discussed.

1. Introduction

Many investigations have been carried out for the flow induced by an oscillating body since Stokes' work [1] relating to spherical and cylindrical pendulums. One of the characteristic features of the flow around an oscillating body is the persistence of a steady streaming motion caused by the nonlinearity of the flow field. The problem of rotational oscillations of a body of revolution has attracted the attention of many researchers for developing a method of measurement of viscosity. The flow field around a torsionally oscillating disk in a viscous fluid was first analyzed by Rosenblat [2] in 1958. He obtained a solution for the velocity field in the form of a power series in ϵ, the ratio of the amplitude of oscillations to the frequency, on the assumption that $\epsilon << 1$ and found the appearance of the steady streaming motion in the second approximation, $O(\epsilon)$. However, he was confronted with the difficulty that the steady streaming solution obtained did not satisfy the boundary condition at infinity. Later, Riley [3] showed the possibility to overcome this difficulty and to obtain a uniformly valid solution by using the method of matched asymptotic expansions.

In the present paper the flow field between a torsionally oscillating and a stationary disk is analyzed in the first place, which has never been undertaken so far, though the case of rotating flow between two coaxial disks has been vigorously investigated, for example, formerly by Batchelor [4] and Stewartson [5], and later by Mellor, Chapple, Stokes [6] and others.

The parameter ϵ is assumed to be very small in the present analysis. The solution for each velocity component is obtained in the form of a power series in ϵ. The solution describing the steady streaming motion appears to the second approximation of $O(\epsilon)$ and satisfies the boundary conditions exactly without any difficulty, as the domain is bounded in the axial direction in the present problem. The unsteady solution to the second-order approximation, though it is out of our main aim, is also obtained. It shows an oscillating motion with frequency twice that of the primary oscillations. The variations of the non-dimensional radial and axial velocities of the steady streaming flow are shown against the axial distance in the figures for a couple of values of the non-dimensional distance between the two coaxial disks. The transverse shearing stresses on the two coaxial disks are also calculated up to the second-order approximation in the ϵ-expansion.

In the last part of this paper we consider the problem of heat transfer in this flow field. The surface temperatures for the oscillating and stationary disks are assumed to be uniform. The analysis is developed on the basis of the energy equation and the temperature field is assumed to depend only on the time and the axial co-ordinate. The solution is expanded in the form of an ascending power series in ϵ and is exactly determined up to the third order. The steady part of the solution in the temperature field consists of terms of $O(\epsilon^0)$ and $O(\epsilon^2)$. The first is due to pure conduction, and the second to the effect of steady streaming. The time-averaged Nusselt numbers on the surfaces of the oscillating and stationary disks are calculated and then the mechanism of laminar heat transfer in the torsionally oscillating flow between the two coaxial disks is discussed.

2. Formulation of the problem

Consider an incompressible viscous flow between two infinite coaxial disks. The upper disk is assumed to be stationary, and the lower disk (placed distance d apart from the upper) has angular velocity $\omega\cos\lambda t$, or, in complex notation, $\omega e^{i\lambda t}$, where ω and λ are the amplitude and frequency of the oscillations. The two disks are kept at uniform temperatures T_d and T_0, respectively.

Now, we take the z-axis coinciding with the co-axis of the disks and introduce the cylindrical polar co-ordinate system (r, ϕ, z). Let u, v, w denote the radial, transverse, axial components of velocity and p and T the pressure and the temperature of the fluid, respectively. Then the appropriate Navier-Stokes equations of motion are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right\}, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left\{\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right\}, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right\}, \tag{3}$$

and the equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

while the energy equation is

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \kappa\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right\}, \tag{5}$$

where ν is the kinematic viscosity, κ the thermal diffusivity and ρ the density of the fluid. The derivatives with respect to ϕ are omitted owing to the symmetry of the problem. The relevant boundary conditions are

$$\left.\begin{array}{l} u = 0, \quad v = r\omega e^{i\lambda t}, \quad w = 0, \\ T = T_0, \end{array}\right\} \text{ at } z = 0, \tag{6}$$

and

$$\left.\begin{array}{l} u = 0, \quad v = 0, \quad w = 0, \\ T = T_d, \end{array}\right\} \text{ at } z = d. \tag{7}$$

We now proceed to seek a similar solution of the form

$$\begin{aligned} &u = r\omega F'(\tau, \eta), \quad v = r\omega G(\tau, \eta), \quad w = -2\omega\sqrt{2\nu/\lambda}\, F(\tau, \eta), \\ &p/\rho = \omega\nu P(\eta) + \frac{1}{2}(a + be^{2i\tau})\,\omega^2 r^2, \; \theta(\tau, \eta) = (T - T_d)/(T_0 - T_d), \\ &\eta = \sqrt{\lambda/2\nu}\, z \quad \text{and} \quad \tau = \lambda t, \end{aligned} \tag{8}$$

where the prime denotes differentiation with respect to η, and a and b are constants to be determined in the solution procedure. Introduction of (8) into the governing equations (1), (2), (3) and (5) yields

$$\frac{\partial F'}{\partial \tau} + \epsilon(F'^2 - 2FF'' - G^2 + a + be^{2i\tau}) = \frac{1}{2}F''', \tag{9}$$

$$\frac{\partial G}{\partial \tau} + 2\epsilon(F'G - FG') = \frac{1}{2}G'', \tag{10}$$

$$P' = -8\epsilon FF' + 2\left(2\frac{\partial F}{\partial \tau} - F''\right), \tag{11}$$

and

$$\frac{\partial \theta}{\partial \tau} - 2\epsilon F\theta' = \frac{1}{2Pr}\theta'', \tag{12}$$

respectively, where $\epsilon = \omega/\lambda$ is the ratio of the amplitude of the oscillations to the frequency, $Pr = \nu/\kappa$ the Prandtl number and $R = d\,(\lambda/2\nu)^{\frac{1}{2}}$ the non-dimensional distance between the two coaxial disks. The boundary conditions (6) and (7) are rewritten as

$$\left.\begin{array}{lll} F' = 0, & G = e^{i\tau}, & F = 0, \\ \theta = 1, & & \end{array}\right\} \text{ at } \quad \eta = 0, \tag{13}$$

and

$$\left.\begin{array}{lll} F' = 0, & G = 0, & F = 0, \\ \theta = 0, & & \end{array}\right\} \text{ at } \quad \eta = R, \tag{14}$$

respectively.

3. Velocity field

Now let us assume that a solution of the equations (9) and (10) can be found by expanding F and G in ascending powers of ϵ:

$$F(\tau, \eta) = F_0(\tau, \eta) + \epsilon F_1(\tau, \eta) + \epsilon^2 F_2(\tau, \eta) \; + \; \ldots\ldots\ldots\ldots , \tag{15}$$

$$G(\tau, \eta) = G_0(\tau, \eta) + \epsilon G_1(\tau, \eta) + \epsilon^2 G_2(\tau, \eta) \; + \; \ldots\ldots\ldots\ldots . \tag{16}$$

The boundary conditions (13) and (14) are then transformed into

$$F_N = F'_N = 0, \quad G_0 = e^{i\tau}, \quad G_{N+1} = 0 \quad \text{at} \quad \eta = 0, \tag{17}$$

and

$$F_N = F'_N = 0, \quad G_N = 0 \quad \text{at} \quad \eta = R, \tag{18}$$

where $N = 0, 1, 2, \ldots\ldots$.

Substituting the series (15) and (16) into the equations (9) and (10) and equating coefficients of like powers of ϵ, we obtain equations for $F_0, G_0, F_1, G_1, \ldots$ and so on, successively. From the lowest power of ϵ, $O(\epsilon^0)$, we obtain the equations for F_0 and G_0 as

$$\frac{\partial F'_0}{\partial \tau} = \frac{1}{2} F'''_0 , \tag{19}$$

$$\frac{\partial G_0}{\partial \tau} = \frac{1}{2} G''_0 . \tag{20}$$

The solutions satisfying the boundary conditions (17) and (18) are readily obtained as

$$F_0 = 0, \tag{21}$$

and

$$G_0 = \frac{e^{i\tau}}{e^{\sqrt{2i}R} - e^{-\sqrt{2i}R}} \{e^{\sqrt{2i}(R-\eta)} - e^{-\sqrt{2i}(R-\eta)}\}, \tag{22}$$

of which, of course, only the real part,

$$\begin{aligned} \mathcal{R}e\,\{G_0\} &= \frac{e^R}{e^{4R} - 2e^{2R}\cos 2R + 1} \\ &\times [(e^{2R} - 1)\cos R\,[e^{(R-\eta)}\cos\{\tau + (R-\eta)\} - e^{-(R-\eta)}\cos\{\tau - (R-\eta)\}] \\ &+ (e^{2R} + 1)\sin R\,[e^{(R-\eta)}\sin\{\tau + (R-\eta)\} - e^{-(R-\eta)}\sin\{\tau - (R-\eta)\}]] \end{aligned} \tag{23}$$

has a physical meaning.

Next, to the second order of approximation, $O(\epsilon)$, we have the equations for F_1 and G_1, the latter of which is of the same form as eq. (20). The solution G_1 satisfying the boundary conditions (17) and (18) is

$$G_1 = 0. \tag{24}$$

On the other hand, the equation for F_1 is

$$\frac{\partial F_1'}{\partial \tau} - \frac{1}{2} F_1''' = G_0^2 - a - be^{2i\tau}. \tag{25}$$

The solution F_1 is assumed to be of the form

$$F_1(\tau, \eta) = f(\eta) + g(\eta)\,e^{2i\tau}, \tag{26}$$

in view of the following form of G_0^2,

$$\begin{aligned} G_0^2 &= \frac{1}{2}\alpha^2(\beta^2 + \gamma^2)\{e^{2(R-\eta)} + e^{-2(R-\eta)} - 2e^{2(R-\eta)i}\} \\ &+ \frac{1}{2}\alpha^2(\beta^2 - 2\beta\gamma i - \gamma^2)\{e^{2(1+i)(R-\eta)} + e^{-2(1+i)(R-\eta)} - 2\}e^{2i\tau}, \end{aligned} \tag{27}$$

where

$$\alpha = \frac{e^R}{e^{4R} - 2e^{2R}\cos 2R + 1}, \tag{28}$$

$$\beta = (e^{2R} - 1)\cos R, \tag{29}$$

and

$$\gamma = (e^{2R} + 1)\sin R. \tag{30}$$

Substitution of the equations (26) and (27) into (25) yields the equations for $f(\eta)$ and $g(\eta)$:

$$f''' = 2a - \alpha^2(\beta^2 + \gamma^2)\{e^{2(R-\eta)} + e^{-2(R-\eta)} - 2e^{2(R-\eta)i}\}, \tag{31}$$

and

$$g''' - 4ig' =$$
$$2b - \alpha^2(\beta^2 - 2\beta\gamma i - \gamma^2)\{e^{2(1+i)(R-\eta)} + e^{-2(1+i)(R-\eta)} - 2\}. \tag{32}$$

The boundary conditions to be imposed on $f(\eta)$ and $g(\eta)$ are

$$f(0) = f(R) = 0, \quad f'(0) = f'(R) = 0, \tag{33}$$

and

$$g(0) = g(R) = 0, \quad g'(0) = g'(R) = 0. \tag{34}$$

These boundary conditions (33) and (34) are also used to determine the unknown constants a and b introduced in (8).

The solution of the equation (31) satisfying the boundary conditions (33) is easily obtained as

$$f'(\eta) = \delta_1\{e^{2(R-\eta)} + e^{-2(R-\eta)} + 2e^{2(R-\eta)i} + A\eta^2 + \Phi_0\eta + \Phi_1\}, \tag{35}$$

and

$$f(\eta) = -\frac{\delta_1}{6}\{3e^{2(R-\eta)} - 3e^{-2(R-\eta)} - 6ie^{2(R-\eta)i} - 2A\eta^3 - 3\Phi_0\eta^2 - 6\Phi_1\eta - 6\Phi_2\}, \tag{36}$$

where the constants δ_1, A and a stand for

$$\delta_1 = -\frac{1}{4}\alpha^2(\beta^2 + \gamma^2), \tag{37}$$

$$A = \frac{3}{R^3}\{(1-R)e^{2R} - (1+R)e^{-2R} - 2(R\cos 2R - \sin 2R) - 4R\}, \tag{38}$$

$$a = -\frac{A}{4}\alpha^2(\beta^2 + \gamma^2), \tag{39}$$

and the integration constants Φ_0, Φ_1 and Φ_2 are, in real notation,

$$\mathscr{R}_e\{\Phi_0\} = \frac{1}{R^2}\{(4R-3)e^{2R} + (4R+3)e^{-2R}$$
$$+ 2(4R\cos 2R - 3\sin 2R) + 8R\}, \tag{40}$$

$$\mathscr{R}_e\,\{\Phi_1\} = -\,(e^{2R} + e^{-2R} + 2\cos 2R), \tag{41}$$

and

$$\mathscr{R}_e\,\{\Phi_2\} = \frac{1}{2}\,(e^{2R} - e^{-2R} + 2\sin 2R). \tag{42}$$

Next, consideration of the equation (32) and the boundary conditions (34) leads to

$$g(\eta) = \frac{\sqrt{2}}{4}\,(1-i)\,\{\Psi_0\,e^{\sqrt{2}(1+i)\eta} - \Psi_1\,e^{-\sqrt{2}(1+i)\eta}\}$$
$$+\frac{\delta_2}{16}\,(1+i)\,\{e^{2(1+i)(R-\eta)} - e^{-2(1+i)(R-\eta)} + 2(1+i)C\eta - 2(1+i)\Psi_2\}, \tag{43}$$

where the integration constants Ψ_0, Ψ_1 and Ψ_2 are expressed as

$$\Psi_0 = \frac{\sqrt{2}\,\delta_2\,i}{8B}\,[\{e^{2(1+i)R} - e^{-2(1+i)R} - 4(1+i)R\}\{1 - e^{-\sqrt{2}(1+i)R}\}$$
$$+\sqrt{2}\,[\{\sqrt{2}\,(1+i)R + 1\}\,e^{-\sqrt{2}(1+i)R} - 1]\,\{e^{2(1+i)R} + e^{-2(1+i)R} - 2\}], \tag{44}$$

$$\Psi_1 = \frac{1}{1 - e^{-\sqrt{2}(1+i)R}}\,[\Psi_0\,\{e^{\sqrt{2}(1+i)R} - 1\} + \frac{\delta_2}{4}\,i\,\{e^{2(1+i)R} + e^{-2(1+i)R} - 2\}] \tag{45}$$

and

$$\Psi_2 = \frac{\sqrt{2}}{\delta_2}\,(1+i)\,(\Psi_1 - \Psi_0) + \frac{1}{4}\,(1-i)\,\{e^{2(1+i)R} - e^{-2(1+i)R}\}, \tag{46}$$

the constants δ_2, B, C and b standing for

$$\delta_2 = \alpha^2\,(\beta^2 - 2\beta\gamma i - \gamma^2), \tag{47}$$

$$B = [\{\sqrt{2}\,(1+i)R - 1\}\,e^{\sqrt{2}(1+i)R} + 1]\,\{e^{-\sqrt{2}(1+i)R} - 1\}$$
$$-[\{\sqrt{2}\,(1+i)R + 1\}\,e^{-\sqrt{2}(1+i)R} - 1]\,\{e^{\sqrt{2}(1+i)R} - 1\}, \tag{48}$$

$$C = \frac{4i}{\delta_2}\,\{\Psi_0\,e^{\sqrt{2}(1+i)R} + \Psi_1\,e^{-\sqrt{2}(1+i)R}\} + 2, \tag{49}$$

and

$$b = \delta_2\,(1 + \frac{1}{2}\,C). \tag{50}$$

Thus, the solution to the second approximation yields the radial and axial components of the velocity and the pressure gradient in the field, giving no effects on the transverse velocity. It consists of two parts, a steady part and an oscillating part with a frequency twice that of the original oscillations. The former expresses the so-called steady streaming motion, to which much attention has been drawn so far. The present solution for $f(\eta)$ or $f'(\eta)$ satisfies exactly the boundary conditions on both disks without such a difficulty as in the case of a single oscillating disk.

Figures 1-3 show the variations of f and f' against η for respective values of $R = 0.1$, 1 and 10. First, comparison of these figures indicates that the larger the non-dimensional distance R is, the more strongly the steady streaming motion occurs. In fact, the magnitude of f or f' for $R = 10$ is about 10^5 times as large as that for $R = 0.1$. Secondly, in these figures, $f'(\eta)$ vanishes at a certain value of η, say η_0, between the two coaxial disks, where the radial velocity vanishes and the axial one becomes maximal. This zero plane is about the mid-plane between the two disks for small R and moves down towards the oscillating disk with increasing R. The steady streaming flows inwards between the stationary disk and this plane, and outwards in the other domain. These phenomena are well-understood when the non-dimensional distance $R = d\,(\lambda/2\nu)^{\frac{1}{2}}$ is considered to have a similar composition to the square root of the Reynolds number.

The transverse shearing stresses on the oscillating and stationary disks are defined by

$$\tau_0 = \rho\nu\left(\frac{\partial v}{\partial z}\right)_{z=0} = \rho\nu r\omega\sqrt{\lambda/2\nu}\,\frac{\partial G}{\partial\eta}\bigg|_{\eta=0},$$

and (51)

$$\tau_R = \rho\nu\left(\frac{\partial v}{\partial z}\right)_{z=R} = \rho\nu r\omega\sqrt{\lambda/2\nu}\,\frac{\partial G}{\partial\eta}\bigg|_{\eta=R}.$$

These are easily evaluated from G_0 only, since $G_1 = 0$, as

$$\tau_0 = -\frac{\rho r\omega\sqrt{\nu\lambda}\,e^R}{e^{4R} - 2e^{2R}\cos 2R + 1}$$
$$\times\Big[(e^{2R} - 1)\cos R\,\{e^R\cos(\tau + R + \frac{\pi}{4}) + e^{-R}\cos(\tau - R + \frac{\pi}{4})\}$$
$$+ (e^{2R} + 1)\sin R\,\{e^R\cos(\tau + R - \frac{\pi}{4}) + e^{-R}\cos(\tau - R - \frac{\pi}{4})\}\Big], \tag{52}$$

and

$$\tau_R = -\frac{2\rho r\omega\sqrt{\nu\lambda}\,e^R}{e^{4R} - 2e^{2R}\cos 2R + 1}$$
$$\times\{(e^{2R} - 1)\cos R\cdot\cos(\tau + \frac{\pi}{4}) + (e^{2R} + 1)\sin R\cdot\cos(\tau - \frac{\pi}{4})\}. \tag{53}$$

In the limiting case of $R \to \infty$, τ_0 reduces to

$$\tau_0 = -\rho r\omega\sqrt{\nu\lambda}\cos(\tau + \frac{\pi}{4}), \tag{54}$$

which is in complete agreement with the result given by Rosenblat for a single oscillating disk, while τ_R tends to zero.

Finally, applying the above-obtained τ_0 in (52), we shall now proceed to calculate the fric-

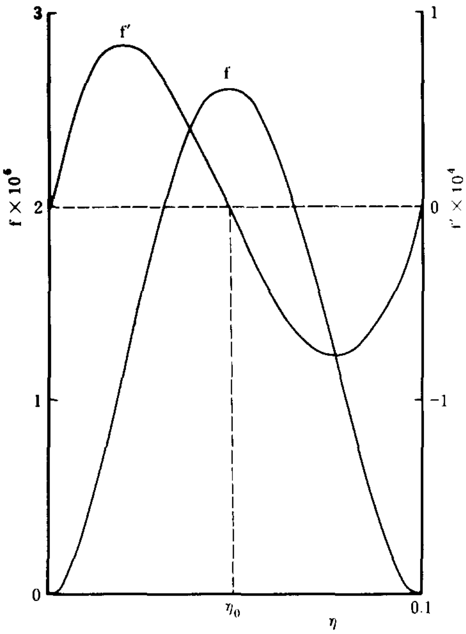

Figure 1. Graphs of f and f' against η in the case of $R = 0.1$.

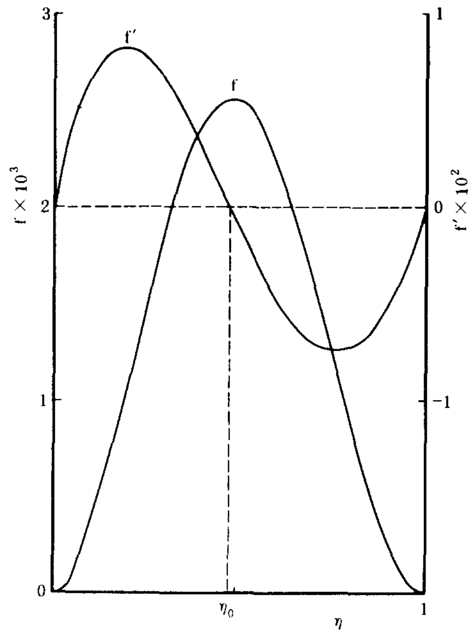

Figure 2. Graphs of f and f' against η in the case of $R = 1$.

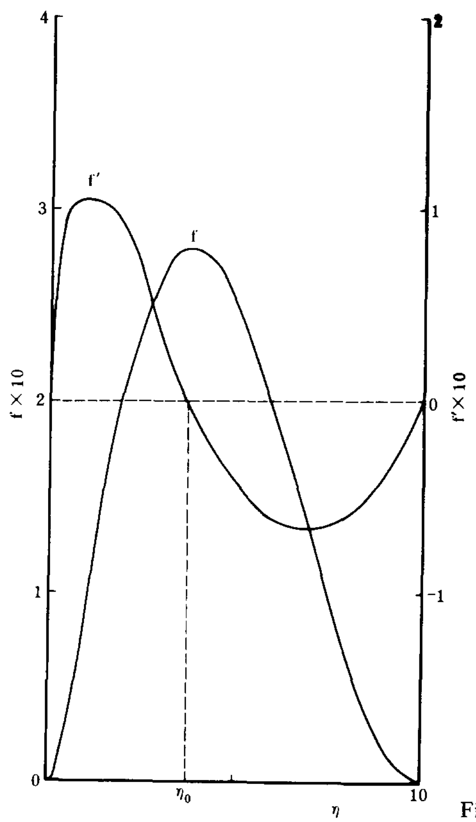

Figure 3. Graphs of f and f' against η in the case of $R = 10$.

tional torque experienced by a finite disk of radius D oscillating torsionally in a viscous fluid between two infinite parallel plates $2R$ apart from one another. This is a sort of model for a viscometer. For the sake of simplicity, the disk is assumed to oscillate in the mid-plane between the two plates. When the radius D is sufficiently large compared with R, edge effects on the torque may be neglected. The torque on both surfaces of the disk, M, is then evaluated by the following integral,

$$M = -4\pi \int_0^D r^2 \tau_0 \, dr. \tag{55}$$

Inserting (52) in τ_0 in the above integral and performing the integration, we have

$$\begin{aligned} M = {} & \frac{\pi D^4 \rho \epsilon \sqrt{\nu \lambda^3}\, e^R}{e^{4R} - 2e^{2R} \cos 2R + 1} \\ & \times [(e^{2R} - 1) \cos R \{e^R \cos(\tau + R + \frac{\pi}{4}) + e^{-R} \cos(\tau - R + \frac{\pi}{4})\} \\ & + (e^{2R} + 1) \sin R \{e^R \cos(\tau + R - \frac{\pi}{4}) + e^{-R} \cos(\tau - R - \frac{\pi}{4})\}], \end{aligned} \tag{56}$$

which, again, tends to the formula given by Rosenblat when $R \to \infty$, and must be more adequate for a viscometer than the latter.

4. Temperature field

The fundamental equation and the boundary conditions for the temperature field are given in (12) and in (13) and (14), respectively. We shall now seek for the solution θ as a power series in ϵ,

$$\theta(\tau, \eta) = \sum_n \epsilon^n \theta_n(\tau, \eta). \tag{57}$$

The solutions θ_0 and θ_1 satisfying the boundary conditions (13) and (14) are readily obtained as

$$\theta_0 = 1 - \frac{\eta}{R}, \tag{58}$$

and

$$\theta_1 = 0. \tag{59}$$

Next, the equation for θ_2 becomes

$$\theta_2'' - 2Pr \frac{\partial \theta_2}{\partial \tau} = \frac{4}{R} Pr F_1. \tag{60}$$

Taking the form of F_1 in (26) into account, we may put

$$\theta_2(\tau, \eta) = \theta_{20}(\eta) + \theta_{21}(\eta) e^{2i\tau}, \tag{61}$$

insertion of which in (60) yields the following equations for θ_{20} and θ_{21},

$$\theta_{20}'' = \frac{4}{R} Pr f, \tag{62}$$

and

$$\theta_{21}'' - 4 i Pr \theta_{21} = \frac{4}{R} Pr g. \tag{63}$$

However, the unsteady part θ_{21} will not be considered here since we are interested only in the steady part of the solution. The solution θ_{20} satisfying the relevant boundary conditions is found to be

$$\theta_{20} = -\frac{Pr}{6R} \delta_1 \{3 e^{2(R-\eta)} - 3 e^{-2(R-\eta)} - 6 \sin 2(R-\eta) - \frac{2}{5} A \eta^5$$
$$- \Phi_0 \eta^4 - 4 \Phi_1 \eta^3 - 12 \Phi_2 \eta^2 + \Phi_3 \eta + \Phi_4 \}, \tag{64}$$

where the constants Φ_0, Φ_1 and Φ_2 are already given in (40)-(42) and the new integration constants Φ_3 and Φ_4 are determined from the boundary conditions (13) and (14) as

$$\Phi_3 = \frac{3}{R}(e^{2R} - e^{-2R}) - \frac{6}{R} \sin 2R + \frac{2}{5} AR^4 + \Phi_0 R^3 + 4 \Phi_1 R^2 + 12 \Phi_2 R, \tag{65}$$

and

$$\Phi_4 = 3(e^{-2R} - e^{2R} + 2 \sin 2R). \tag{66}$$

The stationary part of the temperature gradient on the surface of the oscillating disk is calculated as

$$\theta'(0) = -\frac{1}{R} + \frac{Pr}{R} \delta_1 \epsilon^2 (e^{2R} + e^{-2R} - 2 \cos 2R - \frac{1}{6} \Phi_3) + \ldots, \tag{67}$$

and that on the surface of the stationary one as

$$\theta'(R) = -\frac{1}{R} + \frac{Pr}{R} \delta_1 \epsilon^2 (\frac{A}{3} R^4 + \frac{2}{3} \Phi_0 R^3 + 2 \Phi_1 R^2 + 4 \Phi_2 R - \frac{1}{6} \Phi_3) + \ldots. \tag{68}$$

The Nusselt number Nu is composed of the original dimensional quantities, as

$$Nu = -\frac{d \left(\frac{\partial T}{\partial Z}\right)_{z=0 \text{ or } d}}{T_0 - T_d}, \tag{69}$$

where the subscript $z = 0$ or d should be applied when Nu is concerned with the oscillating disk or the stationary one, respectively. If we take the time-averaged Nusselt number, we have only to substitute the stationary part of the temperature gradient (67) or (68) into the above expression, obtaining

$$\overline{Nu} = -R\left(\frac{d\theta}{d\eta}\right)_{\eta=0 \text{ or } R}$$

$$=\begin{cases} 1 - Pr\,\delta_1\,\epsilon^2\left(e^{2R} + e^{-2R} - 2\cos 2R - \dfrac{1}{6}\Phi_3\right) & \text{for } \eta = 0, \quad (70) \\ 1 - Pr\,\delta_1\,\epsilon^2\left(\dfrac{A}{3}R^4 + \dfrac{2}{3}\Phi_0 R^3 + 2\Phi_1 R^2 + 4\Phi_2 R - \dfrac{1}{6}\Phi_3\right) & \text{for } \eta = R. \quad (71) \end{cases}$$

As is seen in the process of solution of the temperature field, the effect of the oscillating disk appears first to the order of ϵ^2, not to the order of ϵ, through the steady streaming and the induced oscillating flow of a frequency twice the original one.

Figure 4 shows the variations of $\overline{Nu}$ at $\eta = 0$ and R plotted against the parameter ϵ for the respective values of $R = 1, 10, 50$ and 100, the Prandtl number being taken as unity for simplicity's sake. For $R = 1$, little deviations from unity can be seen, since the steady streaming flow is too small, as was seen in Fig. 2, to contribute to an increase of heat transfer, and the heat removed from the hot oscillating disk is all transferred to the cold stationary disk only by pure conduction.

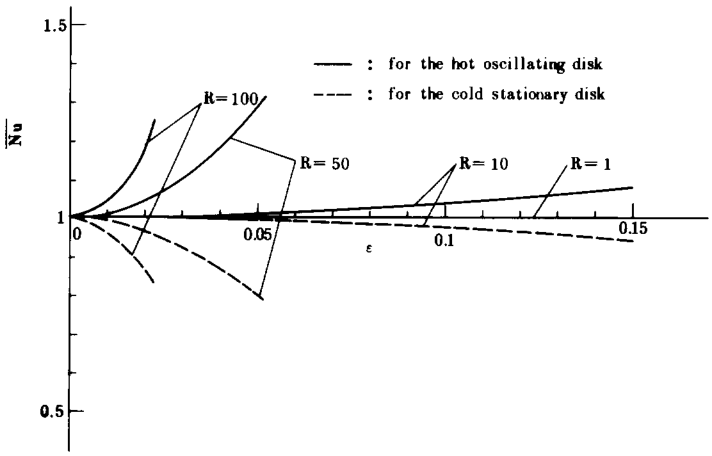

Figure 4. Graphs of $\overline{Nu}$ at the surfaces of the oscillating and stationary disks against ϵ for respective values of $R = 1, 10, 50$ and 100 in the case of $Pr = 1$.

However, the contribution of the steady streaming becomes appreciable with increase of ϵ for $R = 10$ and considerable even at very small ϵ for $R \geq 50$. It should be noted that the larger the values of ϵ and R are, the more heat is removed from the hot oscillating disk and the less heat is received by the cold stationary disk. The difference between these heat amounts is brought away in the radial direction by the steady streaming flow.

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